THE SEMIGROUP APPROACH TO FIRST ORDER QUASILINEAR EQUATIONS IN SEVERAL SPACE VARIABLES

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ABSTRACT

The Cauchy problem for $u_t + \sum_{i=1}^n (\phi_i(u))_{x_i} = 0$ is treated via the theory of semigroups of nonlinear transformations. This treatment requires the development of results concerning the time-independent equation $u + \sum_{i=1}^n (\phi_i(u))_{x_i} = h$ for $h \in L^1(R^n)$, which in turn is studied via the regularized equation

$$u + \sum_{i=1}^n (\phi_i(u))_{x_i} - \varepsilon \Delta u = h.$$

Introduction

This work treats the Cauchy problem (hereafter called (CP)) for the scalar quasilinear equation

$$\text{(DE)} \qquad u_t + \sum_{i=1}^n (\phi_i(u))_{x_i} = 0 \text{ for } t > 0, \ x \in R^n$$

from the point of view of the theory of semigroups of nonlinear transformations. The semigroup approach exhibits the structure of (CP) in a way different from earlier works and provides a new perspective from which to view the problem. The resulting existence theorems go beyond known results. Moreover, as an intermediate step in the development, the existence and uniqueness of certain solutions are established for the equation

$$(1) \qquad u(x) + \sum_{i=1}^n (\phi_i(u(x)))_{x_i} = h(x), \ x \in R^n,$$

for given h.

It is known that any global existence theory for (CP) must involve generalized

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solutions and it turns out that the same is true for (1). On the other hand, the semigroup theory only provides generalized solutions to abstract Cauchy problems. Much of our interest lies in the interplay between the three definitions of "generalized" involved here and in illuminating the scope and nature of the abstract theory. A serious attempt has been made to make Sections 1 and 2 of the text intelligible to the reader who is not familiar with the theory of semigroups of nonlinear transformations.

This investigation was strongly influenced by the penetrating work of Kružkov [13]. Kružkov treats (CP) in the class of bounded and measurable functions and generalizes (DE) to allow the ϕ_i to depend on t and x as well as u. Here we will show how (CP) may be interpreted in $L^1(R^n)$ via the semigroup theory. Quinn [14] makes interesting observations about piecewise continuous solutions of (CP) and semigroups of contractions in $L^1(R^n)$, but does not construct a semigroup or a generator. The introductions and bibliographies of [13], [14] provide further references to the substantial literature dealing with (CP). Also see the recent work [10]. A summary of the semigroup theory is given in [6] and the results used here are proved in [8].

Section 1 of the text contains the necessary definitions, statements of the main results and some preliminary discussion. Section 2 contains the proofs of the main results. Here generalized solutions of (1) are obtained as limits of solutions of the regularized equation

$$u + \sum_{i=1}^{n} (\phi_i(u))_{x_i} - \varepsilon \Delta u = h \tag{2}$$

as $\varepsilon \downarrow 0$. Various results concerning (2) are developed as needed. Section 3 is devoted to the especially simple case $n = 1$ which can be treated in greater generality and simplicity than $n > 1$ and which provides some interesting examples. Section 4 contains comments of various kinds.

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1. Preliminaries and statement of main results

The Cauchy problem (CP) consists of (DE) and the initial condition

$$u(0,x) = u_0(x), \qquad x \in R^n \tag{IC}$$

where u_0 is a given function on R^n. In order to rewrite (CP) in an abstract form, choose a Banach space X of real-valued functions on R^n and regard the function u in (DE) as a map of $[0, \infty)$ into X (the map $t \to u(t, \cdot)$). Let

$$Av = \sum_{i=1}^{n} (\phi_i(v))_{x_i} \tag{1.1}$$

for $v \in D(A)$, where $D(A)$ is a "suitable" subset of X. Then (CP) can be formally rewritten as the abstract Cauchy problem

$$\frac{du}{dt} + Au = 0, \qquad u(0) = u_0. \tag{ACP}$$

The theory of semigroups of nonlinear transformations provides (generalized) solutions to problems of the form (ACP). In order to apply the abstract theory to (ACP), it more than suffices to verify the following very simple conditions on A:

$$R(I + \lambda A) = X \qquad \text{for } \lambda > 0, \tag{1.2}$$

and

$$\|(v + \lambda Av) - (y + \lambda Ay)\| \geqq \|v - y\| \text{ for } \lambda > 0 \text{ and } v, y \in D(A), \tag{1.3}$$

where $\| \quad \|$ denotes the norm in X. It is a real convenience that the theorem concerning generation of semigroups which we will use does not require A to be a function. In greater generality then, let Av be a nonempty subset of X for $v \in D(A)$. Condition (1.2) remains meaningful if we set

$$R(I + \lambda A) = \bigcup\{v + \lambda Av : v \in D(A)\},$$

and (1.3) may be reformulated as

$$\|(v + \lambda w) - (y + \lambda z)\| \geqq \|v - y\| \text{ for } \lambda > 0,\, w \in Av \text{ and } z \in Ay. \tag{1.4}$$

Another formulation of (1.4) is obtained by defining $(I + \lambda A)^{-1}(v + \lambda w) = v$ for $w \in Av$ and $\lambda > 0$. Then $(I + \lambda A)^{-1}$ is a function and (1.4) simply reads

$$\|(I + \lambda A)^{-1}p - (I + \lambda A)^{-1}q\| \leqq \|p - q\| \text{ for } p, q \in R(I + \lambda A) \text{ and } \lambda > 0. \tag{1.5}$$

We will take the term "operator" to include such "multivalued operators". An operator satisfying (1.4) (equivalently, (1.5)) is said to be *accretive* in X.

The following theorem is a special case of the results of [8].

GENERATION THEOREM. *Let X be a Banach space and A be an accretive operator in X such that $R(I + \lambda A) = X$ for $\lambda > 0$. Then for each $\varepsilon > 0$ and $u_0 \in X$ the problem*

$$\begin{cases} \varepsilon^{-1}(u_\varepsilon(t) - u_\varepsilon(t - \varepsilon)) + Au_\varepsilon(t) \ni 0 & \text{for } t \geqq 0 \\ u_\varepsilon(t) = u_0 & \text{for } t < 0 \end{cases} \tag{1.6}$$

has a unique solution $u_\varepsilon(t)$ on $[0, \infty)$. If $u_0 \in \overline{D(A)}$, then $\lim_{\varepsilon \downarrow 0} u_\varepsilon(t)$ exists uni-

formly for t in bounded sets. If $S(t)u_0 = \lim_{\varepsilon\downarrow 0} u_\varepsilon(t)$ *for* $u_0 \in \overline{D(A)}$ *and* $t \geqq 0$, *then* $S(t)$ *is a semigroup of contractions on* $\overline{D(A)}$. *In other words, we have* $S(t): \overline{D(A)} \to \overline{D(A)}$ *for* $t \geqq 0$; $S(t)S(\tau) = S(t+\tau)$ *for* $t, \tau \geqq 0$; $\| S(t)v - S(t)w \| \leqq \| v - w \|$ *for* $t \geqq 0$ *and* $v, w \in D(A)$; $S(0) = I$ *and* $S(t)v$ *is continuous in the pair* (t, v).

If A satisfies the assumptions of the Generation Theorem and S is the associated semigroup, then $S(t)u_0$ can fail to be differentiable for all $t \geqq 0$ even if $u_0 \in D(A)$. However, $S(t)u_0$ provides a notion of a generalized solution of (ACP) (rewritten $0 \in du/dt + Au$, $u(0) = u_0$ if A is multivalued) which agrees with more classical notions of a solution provided that such a solution exists. See [1] and [8].

We will choose $X = L^1(R^n)$ and verify the hypotheses of the Generation Theorem for a generalized version of the A of (1.1). Then the Generation Theorem provides a generalized solution $S(t)u_0$ of (CP) which we show coincides with the generalized solution of Kružkov [13] under suitable hypotheses. The following notation will be used whenever it is meaningful:

$$\begin{aligned}
\phi &= (\phi_1, \cdots, \phi_n): R \to R^n \\
\phi' &= (\phi_1', \cdots, \phi_n') \text{ is the derivative of } \phi \\
\phi(v)_x &= \sum_{i=1}^{n} (\phi_i(v(x)))_{x_i} \text{ if } v: R^n \to R \\
f_x &= (f_{x_1}, \cdots, f_{x_n}) \text{ if } f: R^n \to R \\
ab &= \sum_{i=1}^{n} a_i b_i \text{ if } a, b \in R^n
\end{aligned}$$

$$\operatorname{sign}_0 r = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ -1 & \text{if } r < 0. \end{cases}$$

Our first task is that of defining the A associated with (CP) in $L^1(R^n)$. A will be the closure of A_0, which is defined below:

DEFINITION 1.1. A_0 is the operator in $L^1(R^n)$ defined by: $v \in D(A_0)$ and $w \in A_0(v)$ if $v, w \in L^1(R^n)$, $\phi(v) \in L^1(R^n)$ and

$$\int_{R^n} \operatorname{sign}_0(v(x) - k)\{(\phi(v(x)) - \phi(k))f_x(x) + w(x)f(x)\}dx \geqq 0 \tag{1.7}$$

for every $f \in C_0^\infty(R^n)$ such that $f \geqq 0$ and every $k \in R$.

Equivalent definitions of A_0 are given in the appendix by Brezis. The following lemma helps to motivate Definition 1.1. $C^1(U,V)$ denotes the set of once continuously differentiable maps of U into V. The V or (U,V) will be suppressed when the context makes them obvious. A subscript zero indicates compact support. The normalization

$$\phi(0)=0 \tag{1.8}$$

is assumed throughout this paper.

LEMMA 1.1. *Let $\phi \in C^1$ and A_0 be given by Definition 1.1. If $v \in C_0^1(R^n,R)$, then $v \in D(A_0)$ and $A_0 v = \{\phi(v)_x\}$.*

PROOF. Let $v \in C_0^1(R^n)$. Then, by (1.8), $\phi(v) \in C_0^1(R^n) \subseteq L^1(R^n)$. For $\Phi \in C^1(R,R)$ and $f \in C_0^\infty(R^n)$, integration by parts shows that

$$\begin{aligned}\int_{R^n} (\Phi'(v)\phi(v)_x) f dx &= \int_{R^n} \left(\int_k^{v(x)} \Phi'(s)\phi'(s)ds\right)_x f(x)dx \\ &= -\int_{R^n}\left(\int_k^{v(x)} \Phi'(s)\phi'(s)ds\right) f_x(x)\,dx.\end{aligned} \tag{1.9}$$

Next choose $\Phi(s) = \Phi_l(s-k)$ in (1.9), where

$$\Phi_l(s) = \begin{cases} -s & \text{if } s \leqq -1/l \\ (l/2)s^2 + 1/2l & \text{if } |s| \leqq 1/l \\ s & \text{if } s \geqq 1/l, \end{cases} \tag{1.10}$$

and let $l \to \infty$ to obtain

$$\int_{R^n} \operatorname{sign}_0(v(x)-k)[(\phi(v(x)) - \phi(k))f_x(x) + \phi(v)_x f(x)]\,dx = 0. \tag{1.11}$$

This shows $v \in D(A_0)$ and $\phi(v)_x \in A_0 v$. Finally, assume $v \in D(A_0) \cap L^\infty(R^n)$ and $w \in A_0 v$. Then using successively $k = \|v\|_\infty + 1$ and $k = -(\|v\|_\infty + 1)$ ($\|\ \|_p$ denotes the $L^p(R^n)$ norm) in (1.7) shows that $w = \phi(v)_x$ in the sense of distributions. Hence $A_0 v = \{w\}$, that is, A_0 is single-valued on bounded functions. The proof is complete.

Lemma 1.1 shows that A_0 extends the A of (1.1) on $C_0^1(R^n)$. Indeed, (1.11) holds if $v \in C^1(R^n)$, so if $v \in C^1(R^n) \cap L^1(R^n)$ and $\phi(v) \in L^1(R^n)$, then $\phi(v)_x \in A_0 v$.

Generalized solutions of (1) are defined in terms of A_0. Let A be the closure of A_0, i.e., $v \in D(A)$ and $w \in Av$ if there are sequences $\{v_k\} \subseteq D(A_0)$ and $\{w_k\}$ such that $w_k \in A_0 v_k$ and $v_k \to v$, $w_k \to w$ in $L^1(R^n)$.

DEFINITION 1.2. Let $h \in L^1(R^n)$. Then $u \in L^1(R^n)$ is a generalized solution of

$$u + \phi(u)_x = h \tag{1.12}$$

provided $u \in D(A)$ and $h \in u + Au$.

One of our main results is:

THEOREM 1.1. *Let ϕ be continuous and* $\limsup_{r \to 0} |\phi(r)|/|r| < \infty$. *Then the closure A of A_0 satisfies the assumptions of the Generation Theorem. In particular, $u = (I + A)^{-1}h$ is the unique generalized solution of* (1.12) *for* $h \in L^1(R^n)$.

We will subsequently establish various properties of generalized solutions of (1.12). In particular, we will show that $R(I + A_0) \supseteq L^1(R^n) \cap L^\infty(R^n)$. Thus, if $h \in L^1(R^n) \cap L^\infty(R^n)$, the generalized solution u of (1.12) is defined by setting $v = u$, $w = h - u$ in (1.7) and requiring $\phi(u) \in L^1(R^n)$. Moreover, in this case, $u \in L^\infty(R^n)$ so the end of the proof of Lemma 1.1 shows $u + \phi(u)_x = h$ holds in the sense of distributions.

According to Theorem 1.1 and the Generation Theorem, a semigroup of contractions $S(t)$ is determined by the closure A of A_0. Various properties of this semigroup are listed in the next theorem whose proof consists mainly of establishing analogous results for solutions of (1.12).

THEOREM 1.2. *Let the assumptions of Theorem* 1.1 *hold and S be the semigroup of contractions on* $\overline{D(A)}$ *obtained from A via the Generation Theorem. Let $u, v \in \overline{D(A)}$ and $t \geqq 0$. Then:*

(i) *If $1 \leqq p \leqq \infty$ and $v \in L^p(R^n)$, then $S(t)v \in L^p(R^n)$ and $\| S(t)v \|_p \leqq \| v \|_p$.*

(ii) *If $v \in L^\infty(R^n)$, then*

$$\int_0^T \int_{R^n} \{ |S(t)v(x) - k| f_t + \operatorname{sign}_0(S(t)v(x) - k)[\phi(S(t)v(x)) - \phi(k)] f_x \} dx\,dt \geqq 0$$

for every $f(t,x) \in C_0^\infty((0,T) \times R^n)$ such that $f \geqq 0$ and every $k \in R$ and $T > 0$.

(iii) *If $y \in R^n$, then*

$$\int_{R^n} | S(t)v(x+y) - S(t)v(x) | dx \leqq \int_{R^n} | v(x+y) - v(x) | dx.$$

(iv) *If $h^+ = \max(h, 0)$ for $h \colon R^n \to R$*, then

$$\| (S(t)v - S(t)u)^+ \|_1 \leqq \| (v - u)^+ \|_1.$$

(v)
$$\int_{R^n} S(t)v(x)dx = \int_{R^n} v(x)dx.$$

Kružkov [13] defines a solution of (CP) for $u_0 \in L^\infty(R^n)$ to be a function $u \in L^\infty([0,\infty) \times R^n)$ such that the inequality of (ii) holds with $S(t)v(x)$ replaced by $u(t,x)$ and the initial condition (IC) is satisfied in the sense

$$\lim_{\substack{t \to 0 \\ t \in R^+ - \Omega}} \int_{|x|<K} |u(t,x) - u_0(x)|\, dx = 0 \tag{1.13}$$

for every $K > 0$ and some t-set Ω of measure zero. Kružkov generalizes (DE) to allow ϕ to depend on t and x as well as u. A special case of his results is the existence and uniqueness of solutions of his type for $\phi \in C^1$ when ϕ depends only on u. We do not require $\phi \in C^1$, but we have not been able to show $\overline{D(A)} = L^1(R^n)$ in the generality of Theorems 1.1 and 1.2, although this is obvious if $\phi \in C^1$ (see Lemma 1.1) and can be shown in many other cases. Theorem 1.2 implies $S(t)u_0$ is a solution of Kružkov's type if $u_0 \in \overline{D(A)} \cap L^\infty(R^n)$ ((1.13) holds with $K = \infty$, $\Omega = \emptyset$).

Kružkov's uniqueness proof requires $\phi \in C^1$ (or at least that ϕ be Lipschitz continuous). However it may be modified to establish uniqueness of solutions in the sense (ii) and (1.13) with $K = \infty$ (see [7]). It is interesting to note that if ϕ is not Lipschitz continuous, then (CP) loses its hyperbolic character. More precisely, given t, r, $a > 0$ there will in general not be a number M such that $|u|, |v| \leqq r$ and $u = v$ a.e. on $|x| \leqq M$ implies $S(t)u = S(t)v$ a.e. on $|x| \leqq a$. If $\phi \in C^1$ the hyperbolic nature of (CP) and Theorem 1.2 can easily be used to deduce existence of solutions of Kružkov's type for $u_0 \in L^\infty(R^n)$.

Another interesting point is that we "solve" (CP) for $u_0 \in L^1(R^n)$ and then $S(t)u_0$ is not in $L^\infty(R^n)$, in general. Note that (ii) does *not* make sense for unbounded $S(t)v(x)$ (without extreme restrictions on ϕ), nor does $\phi(v)_x$ have a sense if $\phi(v)$ is not locally integrable. Thus Theorem 1.1 can be regarded as assigning a *meaning* to (CP) for $u_0 \in L^1(R^n)$ and solving the resulting problem at the same time. Moreover, we remark that so-called "entropy" conditions, which acquire a very general form in the notions of generalized solutions of [13], are shown to have, via Theorems 1.1 and 1.2, a stationary sense, i.e., our "entropy" condition is Theorem 1.1 in which "t" does not appear. Theorem 1.2 (iii) implies that if $u_0 \in L^1(R^n)$ has distribution derivatives which are measures of bounded variation, so does $S(t)u_0$. This shows an invariance previously noted in, e.g., [4] and [16]. Theorem 1.2 (iv) shows that $u \geqq v$ implies $S(t)u \geqq S(t)v$, a property well known to be associated with (CP). Also see [12].

2. Proofs of the main results

While Theorem 1.2 appears more complex than Theorem 1.1, it is easily proved once the tools developed to handle Theorem 1.1 are available. The core of the proof of Theorem 1.1 is contained in the various preliminary results given below.

PROPOSITION 2.1. *Let A_0 be given by Definition 1.1. Then A_0 is accretive in $L^1(R^n)$.*

PROOF. We first give a simple tool for demonstrating accretiveness in $L^1(R^n)$.

DEFINITION 2.1. Let $u: R^n \to R$ be measurable. Then sign u is the set of all measurable $v: R^n \to R$ such that $|v(x)| \leqq 1$ a.e. and $v(x)u(x) = |u(x)|$ a.e..

Note that $\operatorname{sign}_0 u \in \operatorname{sign} u$, so sign u is always nonempty.

LEMMA 2.1. *Let $u \in L^1(R^n)$, $v \in L^1(R^n)$ and $\alpha \in \operatorname{sign} u$. If $\int_{R^n} \alpha v dx \geqq 0$, then $\| u + \lambda v \|_1 \geqq \| u \|_1$ for $\lambda \geqq 0$.*

PROOF.

$$\int_{R^n} | u + \lambda v | dx \geqq \int_{R^n} (u + \lambda v)\alpha dx = \int_{R^n} | u | dx + \lambda \int_{R^n} \alpha v dx \geqq \int_{R^n} | u | dx$$

for $\lambda \geqq 0$.

We continue with the proof of Proposition 2.1. Here we borrow strongly from [13]. Let $v \in D(A_0)$, $w \in A_0 v$ and $u \in L^1(R^n)$ be such that $\phi(u) \in L^1(R^n)$. Let $g(x,y) \in C_0^\infty(R^n \times R^n)$, $g \geqq 0$. Set $k = u(y)$, $f(x) = g(x,y)$ in (1.7) and integrate over y. This yields

$$\int_{R^n \times R^n} \operatorname{sign}_0(v(x) - u(y))\{(\phi(v(x)) - \phi(u(y)))g_x(x,y) + w(x)g(x,y)\}dxdy \geqq 0. \tag{2.1}$$

If $u \in D(A_0)$ and $z \in A_0 u$, the inequality symmetric to (2.1) holds (interchange u and v, then z and w, then x and y). Add the two inequalities to obtain

$$\int_{R^n \times R^n} \operatorname{sign}_0(v(x) - u(y))\{(\phi(v(x)) - \phi(u(y)))(g_x + g_y) + (w(x) - z(y))g\}dxdy \geqq 0. \tag{2.2}$$

Now let $\delta \in C_0^\infty(R)$ be non-negative, even and $\int_R \delta = 1$. Let

$$\lambda(x) = \prod_{i=1}^{n} \delta(x_i),\ \lambda_h(x) = h^{-n}\lambda(x/h) \text{ for } h > 0, \tag{2.3}$$

that is, λ_h is a standard smoothing kernel such that convolution with λ_h tends (in various senses) to the identity as $h \downarrow 0$. Let $f \in C_0^\infty(R^n)$ be non-negative and set

$$g(x,y) = f\left(\frac{x+y}{2}\right)\lambda_h\left(\frac{x-y}{2}\right)$$

in (2.2). Set $2\xi = x + y$, $2\eta = x - y$ in the result to yield

$$\int_{R^n}\left[\int_{R^n}(\operatorname{sign}_0(v(\xi+\eta)-u(\xi-\eta))\{(\phi(v(\xi+\eta))-\phi(u(\xi-\eta))f_\xi(\xi)\right. \tag{2.4}$$
$$\left. + (w(\xi+\eta)-z(\xi-\eta))f(\xi)\}d\xi\right]\lambda_h(\eta)d\eta \geqq 0.$$

Denote the integral in [ ] by $I_f(\eta)$. We want to let $h \downarrow 0$ in (2.4). Clearly $I_f(\eta)$ is bounded (since $f \in C_0^\infty(R^n)$), and due to the definition of λ_h we have

$$\liminf_{h\downarrow 0}\int_{R^n} I_f(\eta)\lambda_h(\eta)d\eta \leqq \limsup_{|\eta|\to 0} I_f(\eta). \tag{2.5}$$

Difficulties arise from lack of control of the expression $(\operatorname{sign}_0(v(\xi+\eta)-u(\xi-\eta)))$ $(w(\xi+\eta)-z(\xi-\eta))$ as $|\eta| \to 0$. To handle this we will use the following technical lemma.

Lemma 2.2. *Let $\{\beta_k\}$ be a sequence in $L^1(R^n)$ such that $\beta_k \to \beta$ in $L^1(R^n)$. If $\alpha_k \in \operatorname{sign}\beta_k$, then there is a subsequence $\{\alpha_{k_i}\}$ and $\alpha \in \operatorname{sign}\beta$ such that $\{\alpha_{k_i}\}$ converges to α in the weak-star topology on $L^\infty(R^n)$.*

The proof of this lemma is given in Section 4. Now choose a sequence $\{\eta_k\}$, $|\eta_k| \to 0$, such that $\lim_{k\to\infty} I_f(\eta_k) = \limsup_{|\eta|\to 0} I_f(\eta)$. We use Lemma 2.2 to assume (passing to subsequences if necessary) that

$$\alpha_k(\xi) = \operatorname{sign}_0(v(\xi+\eta_k) - u(\xi-\eta_k))$$

converges weakly-star in $L^\infty(R^n)$ to an element α of $\operatorname{sign}(v(\xi)-u(\xi))$. Then, using (2.4) and (2.5),

$$\limsup_{|\eta|\to 0} I_f(\eta) = \lim_{k\to\infty} I_f(\eta_k) = \int_{R^n}\alpha(\xi)\{(\phi(v(\xi))-\phi(u(\xi)))f_\xi(\xi) \tag{2.6}$$
$$+ (w(\xi)-z(\xi))f(\xi)\}d\xi \geqq 0.$$

In other words, for each non-negative $f \in C_0^\infty(R^n)$, there exists $\alpha \in \operatorname{sign}(v-u)$ (depending, perhaps, on f) such that (2.6) holds. Next choose a function $\kappa \in C_0^\infty(R)$ such that $\kappa \geqq 0$ and $\kappa(s) = 1$ for $|s| \leqq 1$. Set $f(\xi) = \kappa(|\xi|/l)$ in (2.6), let $l \to \infty$ and use Lemma 2.2 in the obvious way to conclude that there is an $\alpha \in \operatorname{sign}(v-u)$ for which

$$\int_{R^n} \alpha(\xi)(w(\xi) - z(\xi))d\xi \geqq 0.$$

Lemma 2.1 now shows

$$\| v - u + \lambda(w - z) \|_1 \geqq \| v - u \|_1 \text{ for } \lambda \geqq 0.$$

Since u, $v \in D(A_0)$ and $w \in A_0 v$, $z \in A_0 u$ were arbitrary, the proof is complete.

It is obvious that the operation of closure preserves accretiveness, so the closure A of A_0 is accretive in $L^1(R^n)$. In order to prove $R(I + \lambda A) = L^1(R^n)$ for $\lambda > 0$, it suffices to show that $R(I + \lambda A_0)$ is dense since when A is closed (i.e., the closure of A is A) and accretive, then $R(I + \lambda A)$ is closed for $\lambda > 0$, as the reader can easily check. We have not yet used *any* assumptions on ϕ (see Proposition 2.1). We will proceed by solving the equation

$$u + \lambda\phi(u)_x - \varepsilon\Delta u = h$$

for $\varepsilon > 0$ in the case $\phi \in C^1$ and ϕ' is bounded. Estimates that are independent of ε and ϕ are obtained which will allow us to let $\varepsilon \downarrow 0$ and to relax the assumptions on ϕ.

$H^k(R^n)$ denotes the Hilbert space of real-valued functions whose distribution derivatives of order at most k lie in $L^2(R^n)$. $H^k(\Omega)$ is defined similarly for open subsets Ω of R^n.

PROPOSITION 2.2. *Let $\phi \in C^1$, ϕ' be bounded and λ, $\varepsilon > 0$. Then for each $h \in L^2(R^n)$ there is a $u \in H^2(R^n)$ such that*

$$u + \lambda\phi(u)_x - \varepsilon\Delta u = h. \tag{2.7}$$

PROOF. Define the operator $B: H^1(R^n) \subseteq L^2(R^n) \to L^2(R^n)$ by $Bu = \phi(u)_x = \sum_{i=1}^n \phi_i'(u)u_{x_i}$ for $u \in H^1(R^n)$. It is clear $u \in H^1(R^n)$ implies $Bu \in L^2(R^n)$ since the assumptions on ϕ imply

$$\| Bu \|_2 \leqq \| \phi'(u) \|_\infty \| u_x \|_2 \leqq K \| u_x \|_2 \leqq K \| u \|_{H^1(R^n)} \tag{2.8}$$

where $K = \sup_{u \in R} |\phi'(u)|$. (We use $|\ \ |$ to denote the Euclidean norm on R^n). The following perturbation result will be used to obtain existence for (2.7).

LEMMA 2.3. *Let H be a Hilbert space, T_1 a non-negative self-adjoint linear operator in H and $T_2: D(T_2) \subseteq H \to H$ be a single-valued operator with the properties $D(T_2) \supseteq D(T_1)$ and*

(i) $$(T_2(u), u) = 0 \textit{ for } u \in D(T_1),$$

(ii) *there is a* $k < 1$ *and a function* $K: R^+ \to R$ *such that* $u \in D(T_1)$ *and* $\| u \| \leqq M$ *implies*

$$\| T_2(u) \| \leqq k \| T_1(u) \| + K(M),$$

(iii) *if* $\Gamma \subseteq D(T_1)$, Γ *and* $T_1\Gamma$ *are bounded then* $T_2: \Gamma \to H$ *is continuous from the relative weak topology on* Γ *into the weak topology on* H.

Then the map $u \to u + T_2(u) + T_1(u)$, $u \in D(T_1)$, *is onto* H.

Lemma 2.3 is essentially a very special case of [2, Theorem 2] and will not be proved here.

In our case we take $H = L^2(R^n)$, $T_2 = \lambda B$, $T_1 = -\varepsilon\Delta$ ($D(T_1) = H^2(R^n)$). We first show $(T_2 u, u) = \lambda(Bu, u) = 0$. One has

$$(Bu,u) = \int_{R^n} (\phi'(u)u_x)u\,dx = \int_{R^n} \psi(u)_x dx \tag{2.9}$$

where $\psi(u) = \int_0^u \phi'(s)s\,ds$. Since ϕ' is bounded, $u \in L^2(R^n)$ implies $\psi(u) \in L^1(R^n)$. The equality $(Bu, u) = 0$ follows from (2.9) and the obvious fact that $v \in L^1(R^n)$ and $v_{x_i} \in L^1(R^n)$ for some i, $1 \leqq i \leqq n$, implies $\int_{R^n} v_{x_i} dx = 0$. (This remains true if $v \in L^p(R^n)$ for some p, $1 < p < \infty$, and $v_{x_i} \in L^1(R^n)$, but is less than obvious in this case.) The estimate (ii) follows from (2.8) since

$$\begin{aligned} \lambda \| B(u) \|_2 &\leqq \lambda K \| u_x \|_2 \leqq \lambda K(\gamma^2 \| \Delta u \|_2^2 + (1/\gamma)^2 \| u \|_2^2)^{\frac{1}{2}} \\ &\leqq \lambda K \gamma (1/\varepsilon) \| \varepsilon \Delta u \|_2 + (\lambda K/\gamma) \| u \|_2 \end{aligned}$$

holds for every $\gamma > 0$. Simply choose $0 < \gamma < \varepsilon/\lambda K$. For the condition (iii), let $\{u_l\}$ be a sequence in $H^2(R^n)$ for which $\{u_l\}$ and $\{-\varepsilon\Delta u_l\}$ are bounded in $L^2(R^n)$. Then $\{u_l\}$ is bounded in $H^2(R^n)$. Let $u_l \rightharpoonup u$ (weakly) in $L^2(R^n)$. Since $\{Bu_l\}$ is bounded, the proof will be complete if we show $Bu_l \rightharpoonup v$ in $L^2(R^n)$ implies $v = Bu$. However, $\{u_l\}$ being bounded in $H^2(R^n)$ implies it is strongly precompact in $H^1(\{x \in R^n: |x| \leqq K\})$ for each K. Thus there is a subsequence $\{u_{l_i}\}$ with the property $u_{l_i} \rightharpoonup u$ in $L^2(R^n)$ and $u_{l_i} \to u$ in $H^1(\Omega)$ for each bounded $\Omega \subseteq R^n$. Hence $Bu_{l_i} \rightharpoonup v$ and $Bu_{l_i} = \phi'(u_{l_i})u_{l_i,x}$ imply $v = Bu$. The hypotheses of Lemma 2.3 have been verified, and proof of Proposition 2.2 is complete. Later we will show the solution of (2.7) is unique for $h \in L^1(R^n) \cap L^2(R^n)$. The next lemma collects some a priori estimates.

LEMMA 2.4. *Let the assumptions of Proposition* 2.2 *hold. Let* $u \in H^2(R^n)$ *satisfy* (2.7) *where* $h \in L^p(R^n)$ *for some* p, $1 \leqq p \leqq \infty$. *Then* $u \in L^p(R^n)$ *and*

$$\| u \|_p \leqq \| h \|_p.$$

PROOF. We treat the case $1 \leqq p < \infty$ first. Choose a sequence $\alpha_{p,l}$ of monotone, Lipschitz continuous, piecewise smooth and odd functions which satisfy

$$\text{(2.10)}\qquad \begin{aligned} &\text{(i)} \quad \lim_{l\to\infty} \alpha_{p,l}(r) = |r|^{p-1}\operatorname{sign}_0 r \quad \text{for} \quad r \in R \\ &\text{(ii)} \quad |r|^{p-1} \geqq |\alpha_{p,l}(r)| \qquad\quad \text{for} \quad r \in R. \end{aligned}$$

The property (2.10) may be restated as

$$\text{(2.11)}\qquad r\alpha_{p,l}(r) \geqq |\alpha_{p,l}(r)|^q, \qquad q = p/(p-1)$$

if $p > 1$. Multiplying (2.7) by $\alpha_{p,l}(u)$ one finds

$$\text{(2.12)}\qquad \begin{aligned} \int_{R^n} (u\alpha_{p,l}(u) + \lambda\phi(u)_x\alpha_{p,l}(u) \; - \; \varepsilon(\Delta u)\alpha_{p,l}(u))dx \; &= \int_{R^n} h\alpha_{p,l}(u)dx \\ &\leqq \; \| h \|_p \| \alpha_{p,l}(u) \|_q \end{aligned}$$

where the right-hand side is finite by (2.10) (ii) (or 2.11) since u, $\alpha_{p,l}(u) \in L^2(R^n)$. The contribution of each term on the left is estimated below:

$$\text{(2.13)}\qquad \int_{R^n} u\alpha_{p,l}(u)dx \; \geqq \int_{R^n} |\alpha_{p,l}(u)|^q dx$$

by (2.10) (ii) and the oddness of u.

$$\text{(2.14)}\qquad \int_{R^n} (\Delta u)\alpha_{p,l}(u)dx \; = \; - \int_{R^n} \alpha'_{p,l}(u)|u_x|^2 dx \leqq 0$$

since $\alpha_{p,l}$ is monotone.

$$\text{(2.15)}\qquad \int_{R^n} \phi(u)_x\alpha_{p,l}(u)dx \; = \int_{R^n} \left(\int_0^u \alpha_{p,l}(s)\phi'(s)ds\right)_x dx = 0$$

since $\int_0^u \alpha_{p,l}(s)\phi'(s)ds \in L^1(R^n)$, because ϕ' is bounded and $|\alpha_{p,l}(s)| \leqq \text{cons.}\ |s|$. Using (2.13)–(2.15) in (2.12) yields

$$\int_{R^n} |\alpha_{p,l}(u)|^q dx \leqq \| h \|_p \| \alpha_{p,l}(u) \|_q$$

or

$$\left(\int_{R^n} |\alpha_{p,l}(u)|^q dx\right)^{1/p} \leqq \| h \|_p.$$

Since $|\alpha_{p,k}(u)|^q \to |u|^p$ as $k \to \infty$, the result follows from Fatou's Lemma (the reader may want to check that $p = 1$ causes no trouble above).

We use a device of Stampacchia [15] to treat $p = \infty$. If $M \geqq h^+$ a.e., then subtracting M from both sides of (2.7) and multiplication by $(u - M)^+$ yields

$$(u-M)(u-M)^+ + \lambda\phi(u)_x(u-M)^+ - \varepsilon(\Delta u)(u-M)^+ = (h-M)(u-M)^+ \leqq 0.$$

Now $\int_{R^n}\phi(u)_x(u-M)^+dx = 0$ and $\int_{R^n}\Delta u(u-M)^+dx \leqq 0$ as before, so $\int_{R^n}(u-M)(u-M)^+dx \leqq 0$, which implies $(u-M)^+ \leqq 0$ or $u \leqq M$ a.e.. A similar estimate shows that if $M \geqq h^- = \max(0,-h)$ then $-M \leqq u$ a.e.. Since we can take $M = \| h \|_\infty$, the proof is complete.

REMARK 2.1. The proof shows $-\| h^- \|_\infty \leqq u \leqq \| h^+ \|_\infty$.

The next estimate establishes uniqueness of solutions of (2.7) for $h \in L^2(R^n \cap L^1(R^n)$ and is the heart of our development.

PROPOSITION 2.3. *Let the assumptions of Proposition 2.2 hold and $u, v \in H^2(R^n)$ satisfy*

$$u + \lambda\phi(u)_x - \varepsilon\Delta u = h$$

$$v + \lambda\phi(v)_x - \varepsilon\Delta v = g$$

where $\varepsilon > 0$. If $h, g \in L^1(R^n)$, then

$$\|(u-v)^+\|_1 \leqq \|(h-g)^+\|_1.$$

PROOF. Define ψ_l by $\psi_l' = \Phi_l'^+$, $\psi_l(0) = 0$, where Φ_l is given by (1.10). Let $1 \geqq f \geqq 0, f \in C_0^\infty(R^n)$ and $w = u - v$. Then

$$w + \lambda(\phi(u) - \phi(v))_x - \varepsilon\Delta w = h - g.$$

Multiplying the above by $\psi_l'(w)f$ and integration yields

$$\int_{R^n}\{w\psi_l'(w)f + \lambda(\phi(u)-\phi(v))_x\psi_l'(w)f - \varepsilon(\Delta w)\psi_l'(w)f\}dx \leqq \|(h-g)^+\|_1 \tag{2.16}$$

since $0 \leqq \psi_l' f \leqq 1$. Now $\psi_l'(w) \in H^1(R^n)$, $\psi_l'', f \geqq 0$ so

$$\begin{aligned}\int_{R^n}(\Delta w)\psi_l'(w)fdx &= -\int_{R^n}\psi_l''(w)|w_x|^2fdx - \int_{R^n}\psi_l'(w)w_xf_xdx \\ &= -\int_{R^n}\psi_l''(w)|w_x|^2fdx + \int_{R^n}\psi_l(w)\Delta fdx \leqq \int_{R^n}\psi_l(w)\Delta fdx.\end{aligned}$$

Letting $l \to \infty$ yields

$$\limsup_{l\to\infty}\int_{R^n} -(\Delta w)\psi_l'(w)fdx \geqq -\int_{R^n} w^+\Delta fdx, \tag{2.17}$$

where we use that $w \in L^1(R^n)$ by Lemma 2.4. Next,

$$\begin{aligned}\int_{R^n}(\phi(u)-\phi(v))_x\psi_l'(w)fdx = &-\int_{\Omega_l}\psi_l''(w)((\phi(u)-\phi(v))w_xfdx \\ &-\int_{R^n}\psi_l'(w)(\phi(u)-\phi(v))f_xdx,\end{aligned}$$

where $\Omega_l = \{x : |u(x) - v(x)| \leqq 1/l\}$ and we used that $\psi_l''(w) = 0$ off Ω_l. For the first term we note $|\psi_l''(w)(\phi(u) - \phi(v))| \leqq l K|u - v| \leqq K$ on Ω_l where $K \geqq |\phi'|$. Hence, by the dominated convergence theorem,

$$\limsup_{l\to\infty} \left| \int_{\Omega_l} \psi_l'(w)(\phi(u) - \phi(v))w_x f dx \right| \leqq K \int_{\Omega} |w_x| f dx$$

where $\Omega = \cap_l \Omega_l$. But $u = v$ on Ω implies $w_x = (u - v)_x = 0$ a.e. on Ω (see [15, appendix]) so the integral on the right is zero. Thus

$$\text{(2.18)} \quad \limsup_{l\to\infty} \int_{R^n} \psi_l'(w)(\phi(u) - \phi(v))_x f dx \geqq - \int_{R^n} |(\phi(u) - \phi(v))f_x| dx.$$

Using (2.18) and (2.17) in (2.16) and letting $l \to \infty$ yields

$$\text{(2.19)} \int_{R^n} w^+ f dx - \varepsilon \int_{R^n} w^+ \Delta f dx - \int_{R^n} |(\phi(u) - \phi(v))f_x| dx \leqq \|(g - h)^+\|_1$$

for every $f \in C_0^\infty(R^n)$, $1 \geqq f \geqq 0$. Choose a function $\kappa \in C_0^\infty(R)$ such that $1 \geqq \kappa \geqq 0$ and $\kappa(s) = 1$ for $|s| \leqq 1$. Set $f(x) = \kappa(\|x\|/l)$ in (2.19) and let $l \to \infty$ (using that $w = u - v$ and $|\phi(u) - \phi(v)|$ are in $L^1(R^n)$) to obtain

$$\int_{R^n} w^+ dx = \int_{R^n} (u - v)^+ dx \leqq \|(g - h)^+\|_1.$$

The proof is complete.

The next corollary is an immediate consequence of Proposition 2.3.

COROLLARY 2.1. *Let the assumptions of Proposition* 2.2 *be satisfied. Let* $u, v \in H^2(R^n)$ *and*

$$u + \lambda\phi(u)_x - \varepsilon\Delta u = h$$

$$v + \lambda\phi(v)_x - \varepsilon\Delta v = g$$

where $h,\ g \in L^1(R^n)$. *Then* $\|u - v\|_1 \leqq \|h - g\|_1$. *Moreover, if* $g \geqq h$ *a.e., then* $v \geqq u$ a.e..

With the next result, the proof of Theorem 1.1 is essentially complete.

COROLLARY 2.2. *Let* ϕ *be continuous and* $\limsup_{r\to 0} |\phi(r)|/|r| < \infty$. *Let* A_0 *be given by Definition* 1.1. *Then* $R(I + \lambda A_0) \supseteq L^1(R^n) \cap L^\infty(R^n)$ *for* $\lambda > 0$. *Let* $T_\lambda : L^1(R^n) \cap L^\infty(R^n) \to L^1(R^n)$ *be the restriction of* $(I + \lambda A_0)^{-1}$ *to* $L^1(R^n) \cap L^\infty(R^n)$. *If* $h, g \in L^1(R^n) \cap L^\infty(R^n)$, *then the following statements hold*:

(i) *If* $1 \leqq p \leqq \infty$ *and* $h \in L^p(R^n)$, *then* $T_\lambda h \in L^p(R^n)$ *and* $\|T_\lambda h\|_p \leqq \|h\|_p$.

(ii) $-\|h^-\|_\infty \leqq T_\lambda h \leqq \|h^+\|_\infty$ a.e..

(iii) $\| (T_\lambda h - T_\lambda g)^+ \|_1 \leqq \| (h - g)^+ \|_1$.

(iv) $\int_{R^n} T_\lambda g(x)\,dx = \int_{R^n} g(x)\,dx$.

Moreover, T_λ commutes with translations.

PROOF. Choose a sequence $\{\phi_l\}_{l=1}^{\infty}$ of C^1 functions such that $\phi_l(0) = 0$, ϕ_l' is bounded and $\{\phi_l\}$ converges to ϕ uniformly on compact sets. Given $\lambda > 0$, define $T_{\lambda,l}$: $L^1(R^n) \cap L^\infty(R^n) \to L^1(R^n) \cap L^\infty(R^n)$ by $T_{\lambda,l}\, h = u$ if $u \in H^2(R^n)$ and

$$u + \lambda\phi_l(u)_x - (1/l)\Delta u = h.$$

$T_{\lambda,l}$ is well-defined by Propositions 2.2, 2.3 and Lemma 2.4 and it has the properties claimed for T_λ in Corollary 2.2 by virtue of Lemma 2.4, Remark 2.1, Proposition 2.3 and the obvious fact that $T_{\lambda,l}$ commutes with translations. Let $h \in L^1(R^n) \cap L^\infty(R^n)$ and $u_l = T_{\lambda,l}h$. By Corollary 2.1 and the translation invariance of $T_{\lambda,l}$ we have

$$\int_{R^n} | u_l(x+y) - u_l(x) |\, dx \leqq \int_{R^n} | h(x+y) - h(x) | dx \tag{2.20}$$

for $y \in R^n$. The estimate (2.20) and $\| u_l \|_1 \leqq \| h \|_1$ imply that $\{u_l\}$ is precompact in $L^1_{loc}(R^n)$. Thus there is a subsequence $\{u_{l(i)}\}$ of $\{u_l\}$ which converges a.e. and in $L^1_{loc}(R^n)$ to a limit $u \in L^1(R^n)$. We denote this convergence by $\twoheadrightarrow$, $u_{l(i)} \twoheadrightarrow u$. Let $f \in C_0^\infty(R^n)$ and Φ: $R \to R$ have a piecewise continuous second derivative. Multiply the equation satisfied by u_l by $\Phi'(u_l)f$. Integration over R^n and some integration by parts yields

$$\begin{aligned} \int_{R^n} \{ & u_l\Phi'(u_l)f - \lambda(\Phi'(u_l)\phi_l(u_l) - \Phi(k)\phi_l(k))f_x \\ & + \lambda\left(\int_k^{u_l} \Phi''(s)\phi_l(s)\,ds\right) f_x + (1/l)\,(\Phi''(u_l)\,|\,u_{lx}|^2 f \\ & - \Phi(u_l)\Delta f)\}dx = \int_{R^n} h\Phi'(u_l)f\,dx \end{aligned}$$

for every $k \in R$. The identity

$$\Phi'(u_l)\,\phi_l(u_l)_x = (\Phi'(u_l)\phi_l(u_l) - \Phi'(k)\phi_l(k) - \int_k^{u_l} \Phi''(s)\phi(s)ds)_x$$

was used in integrating by parts. Assuming $\Phi'' \geqq 0$ and $f \geqq 0$, the term involving $\Phi''(u_l)|\,u_{lx}|^2 f$ is non-negative. Moreover, $\| u_l \|_\infty \leqq \| h \|_\infty$ so $\int_{R^n} \Phi(u_l)\Delta f dx$ is bounded in l. Letting l tend to ∞ through the subsequence $\{l(i)\}$ and using the convergences $u_{l(i)} \twoheadrightarrow u$ and $\phi_l \to \phi$ uniformly on compact sets yields

$$\int_{R^n} \{u\Phi'(u)f - \lambda(\Phi'(u)\phi(u) - \Phi'(k)\phi(k))f_x + \lambda\left(\int_k^u \Phi''(s)\phi(s)ds\right) f_x\}dx \leqq \int_{R^n} h\Phi'(u)f dx$$

for $f \in C_0^\infty(R^n)$, $f \geqq 0$, $k \in R$, $\Phi'' \geqq 0$. Next choose $\Phi(s) = \Phi_l(s-k)$ where Φ_l is given by (1.10) and let $l \to \infty$. Since

$$\int_k^u \Phi''_l(s)\phi(s)ds \to \operatorname{sign}_0(u-k)\phi(k)$$

this yields

$$\int_{R^n} \operatorname{sign}_0(u-k)\{uf - \lambda(\phi(u) - \phi(k))f_x - hf\}dx \leqq 0$$

for every $k \in R$ and non-negative f in $C_0^\infty(R^n)$. (Notice that only continuity of ϕ has been used so far.) Since $\| u \|_\infty \leqq \| h \|_\infty$, and $\limsup_{r\to 0} |\phi(r)|/|r| < \infty$ $\phi(u) \in L^1(R^n)$. According to Definition 1.1, $\lambda^{-1}(h-u) \in A_0 u$, so $h \in u + \lambda A_0 u$. Since u is unique by the accretiveness of A_0, it follows that $\lim_{l\to\infty} T_{\lambda,l}h = T_\lambda h$ holds with convergence in $L^1_{loc}(R^n)$. The assertions (i), (ii) and (iii) of Corollary 2.2 are known for $T_{\lambda,l}$ and are preserved under $L^1_{loc}(R^n)$ convergence, so it remains to establish (iv). If $u = T_\lambda g$, $u \in L^\infty(R^n)$ and $A_0 u = \{\phi(u)_x\}$ by the proof of Lemma 1.1. Thus $\phi(u) \in L^1(R^n)$ and $\phi(u)_x \in L^1(R^n)$, which implies $\int_{R^n}\phi(u)_x dx = 0$ (a similar observation is used in proving Lemma 2.3, etc.). Since $u + \lambda\phi(u)_x = g$, the proof is complete.

Remark 2.2. A result of [3] shows that (ii) and (iii) of Corollary 2.2 imply $\int_{R^n} j(T_\lambda h)dx \leqq \int_{R^n} j(h)dx$ for any non-negative lower semicontinuous convex function $j: R \to R$ such that $j(0) = 0$. Thus (i) is a consequence of (ii) and (iii).

Proof of Theorem 1.1. We remark again that the closure A of A_0 is accretive since A_0 is accretive. Moreover, if $h \in L^1(R^n)$, $\{h_k\} \subseteq L^1(R^n) \cap L^\infty(R^n)$ and $h_k \to h$, then $\{T_\lambda h_k\}$ is Cauchy since T_λ is a contraction. Let $w_k = \lambda^{-1}(h_k - T_\lambda h_k)$, so $w_k \in A_0 T_\lambda h_k$ and $\{w_k\}$ is Cauchy. If $T_\lambda h_k \to v$, $w_k \to w$, then $w \in Av$ and $h = v + \lambda w \in R(I + \lambda A)$.

Proof of Theorem 1.2. The solution $u_\varepsilon(t)$ of (1.6) is given by $u_\varepsilon(t) = (I + \varepsilon A)^{-[t/\varepsilon]-1}u_0$ where $[t/\varepsilon]$ is the greatest integer in t/ε. Since $\lim_{\varepsilon\downarrow 0} u_\varepsilon(t) = S(t)u_0$ with convergence in $L^1(R^n)$, (i), (iii), (iv), (v) hold for $S(t)$ if they hold for $(I + \lambda A)^{-1}$ in place of $S(t)$. However, these relations have been established for T_λ ((iii) follows from the translation invariance) and therefore also hold in general. For example, if $v \in L^1(R^n) \cap L^p(R^n)$ choose $\{v_k\} \subseteq L^1(R^n) \cap L^\infty(R^n)$ such that $\| v_k \|_p \leqq \| v \|_p$ and $v_k \to v$ in $L^1(R^n)$. Then $T_\lambda v_k \to (I + \lambda A)^{-1}v$ as above and

$$\left\| (I + \lambda A)^{-1} v \right\|_p \leqq \liminf_{k \to \infty} \left\| T_\lambda v_k \right\|_p \leqq \left\| v \right\|_p.$$

The inequality (ii) remains to be proved. Let $v \in L^1(R^n) \cap L^\infty(R^n)$ and $u_\varepsilon(t)$ satisfy

$$\text{(2.21)} \qquad \begin{cases} \varepsilon^{-1}(u_\varepsilon(t) - u_\varepsilon(t - \varepsilon)) + A_0 u_\varepsilon(t) = 0 & t \geqq 0 \\ u_\varepsilon(t) = v & t < 0. \end{cases}$$

Then $u_\varepsilon(t) = (I + \varepsilon A_0)^{-[t/\varepsilon]-1} v$ for $t \geqq 0$, and $\left\| u_\varepsilon(t) \right\|_p \leqq \left\| v \right\|_p$ for $p = 1, \infty$. Let $u_\varepsilon(t,x) = u_\varepsilon(t)(x)$. By Definition (1.1),

$$\text{(2.22)} \qquad \begin{aligned} \int_{R^n} &\{\operatorname{sign}_0(u_\varepsilon(t,x) - k)(\phi(u_\varepsilon(t,x)) - \phi(k)) f_x(t,x) \\ &+ \varepsilon^{-1}(u_\varepsilon(t-\varepsilon,x) - u_\varepsilon(t,x)) \operatorname{sign}_0(u_\varepsilon(t,x) - k) f(t,x)\} dx \geqq 0 \end{aligned}$$

for every $k \in R$ and non-negative $f \in C_0^\infty((0,T) \times R^n)$. Let $h_\varepsilon(t,x) = (u_\varepsilon(t,x) - k)$ $\operatorname{sign}_0(u_\varepsilon(t,x) - k) = | u_\varepsilon(t,x) - k |$. Notice that

$$\text{(2.23)} \qquad \begin{aligned} &(u_\varepsilon(t-\varepsilon,x) - u_\varepsilon(t,x)) \operatorname{sign}_0(u_\varepsilon(t,x) - k) \\ &= (u_\varepsilon(t-\varepsilon,x) - k) \operatorname{sign}_0(u_\varepsilon(t,x) - k) - (u_\varepsilon(t,x) - k) \operatorname{sign}_0(u_\varepsilon(t,x) - k) \\ &\leqq h_\varepsilon(t-\varepsilon,x) - h_\varepsilon(t,x). \end{aligned}$$

Using (2.23) and integrating (2.22) over $0 \leqq t \leqq T$ yields

$$\text{(2.24)} \qquad \begin{aligned} \int_0^T \int_{R^n} &\{\operatorname{sign}_0(u_\varepsilon(t,x) - k)(\phi(u_\varepsilon(t,x)) - \phi(k)) f_x(t,x) \\ &+ \varepsilon^{-1}(h_\varepsilon(t-\varepsilon,x) - h_\varepsilon(t,x)) f(t,x)\} \, dxdt \geqq 0. \end{aligned}$$

Now

$$\begin{aligned} &\varepsilon^{-1} \int_0^T \int_{R^n} \{(h_\varepsilon(t-\varepsilon,x) - h_\varepsilon(t,x)) f(t,x)\} dxdt \\ &= \varepsilon^{-1} \left(\int_0^\varepsilon \int_{R^n} h(t-\varepsilon,x) f(t,x) dxdt - \int_{T-\varepsilon}^T \int_{R^n} h_\varepsilon(t,x) f(t,x) dxdt \right) \\ &+ \int_\varepsilon^{T-\varepsilon} \int_{R^n} h_\varepsilon(t,x)(\varepsilon^{-1})(f(t+\varepsilon,x) - f(t,x)) dxdt. \end{aligned}$$

The first and second integrals vanish for ε small enough since f is in $C_0^\infty((0,T) \times R^n)$. The convergence, $u_\varepsilon(t,x) \to S(t)v(x)$ in $L^1(R^n)$, uniformly in t as $\varepsilon \to 0$ implies the third term tends to

$$\int_0^T \int_{R^n} | S(t)v(x) - k | f_t(t,x)dx\,dt$$

as $\varepsilon \downarrow 0$. Hence Theorem 1.2 (ii) follows from letting $\varepsilon \downarrow 0$ in (2.24).

3. A closer look at A when $n = 1$

The exposition in this section is brief and assumes some familiarity with (DE) for $n = 1$ and with accretive operators. A more detailed discussion is supplied in [7]. There is a particular case of (CP) which is especially simple for us, but which is not singled out by other approaches.

THEOREM 3.1. *Let $\phi: R \to R$ be continuous and strictly monotone. Let $Av = \phi(v)_x$ for $v \in D(A) = \{u : u \in L^1(R) \cap L^\infty(R)$ and $\phi(u)_x \in L^1(R)\}$. Then A is accretive and $R(I + \lambda A) \supseteq L^1(R) \cap L^\infty(R)$ for $\lambda > 0$. If, moreover, $\phi(R) = R$, then A is closed.*

SKETCH OF PROOF. Let $u \in D(A)$. Then $\phi(u)_x \in L^1(R)$ implies $\phi(u)$ is continuous and has limits at $\pm\infty$. Hence $u = \phi^{-1}(\phi(u))$ has the same properties, and $u \in L^1(R)$ implies u tends to zero at $\pm\infty$. Now if u, $v \in D(A)$

$$\int_{-\infty}^{\infty} (\phi(u) - \phi(v))_x \operatorname{sign}_0(u - v)dx = 0$$

since the integral may be written as a sum of integrals over the countable number of components of $\{x : v(x) \neq u(x)\}$. Each such integral is evidently zero. It is easy to see that A is closed if $\phi(R) = R$. To show, e.g., $R(I + A) \supseteq L^1(R) \cap L_\infty(R)$, it suffices to show $R(I + A) \supseteq C_0^\infty(R)$ and use suitable limiting arguments. Let $h \in C_0^\infty(R)$. If, e.g., ϕ is increasing and $h = 0$ on $(-\infty, a]$, solve

$$u(a) = 0, \; u + \phi(u)_x = h \text{ for } x > a.$$

The existence of such a u is a simple problem in ordinary differential equations (change variables by $w = \phi(u)$) as is the fact that $u \in D(A)$ (set equal to 0 on $(-\infty, a]$). The sketch is complete. Note that $D(A)$ is dense in $L^1(R)$ in this case.

This "special case" turns out to be quite general, as the next remark of P. D. Lax shows. Assume we want to solve

$$u_t + \phi(u)_x = 0, \; u(0,x) = u_0(x)$$

where ϕ is not monotone, but $u_0 \in L^\infty(R)$. Then set $v(t,x) = u(t, x - ct)$. The problem for v is

$$v_t + (\phi(v) + cv)_x = 0,\ v(0,x) = u_0(x)\,.$$

If c can be chosen such that $\phi(s) + cs$ is monotone on $\{s: |s| \leqq \|u_0\|_\infty + 1\}$, we are actually reduced to the case just treated (note that the behavior of $\phi(s)$ for $|s| > \|u_0\|_\infty$ plays no role, as in Section 2.)

Even though the case of monotone ϕ is nice from the generator point of view, it can exhibit the following pathology:

EXAMPLE 3.1. Consider $\phi(u) = u^3$. Let A be as in Theorem 3.1, and $S(t)$ the associated semigroup. If $u_0 \in C_0^\infty(R)$, $u_0 \geqq 0$ and u_0 is not identically zero, then $S(t)u_0 \notin D(A)$ for some $t > 0$. To see this, note that $S(t)u_0(x) = u(t,x)$ where $u(t,x)$ is the classical solution of

$$u_t + (u^3)_x = 0,\ u(0,x) = u_0(x),$$

the equality holding while the classical solution exists. Analysis by the method of characteristics shows there is a positive t_0 such that $u(t,x)$ exists for $0 \leqq t < t_0$, $x \in R$ but the limit $\lim_{t \uparrow t_0} u(t,x)$ (taken, e.g., in $L^1(R)$) is not continuous. Hence $S(t_0)u_0$ is not in $D(A)$. Other examples of this kind are known. However they are either uninteresting from the point of view of partial differential equations or depend on a "poor" choice of X.

We leave the monotone ϕ case now to observe other phenomena.

EXAMPLE 3.2. Let $\phi(u) = \frac{1}{2}u^2$. Then A, defined as in Theorem 3.1, is *not* accretive. In particular, "weak" solutions of $u + (u^2)_x = h$ are, in general, not unique. To see this, let $u(x) = -\max(x+1, 0)$ for $x < 0$ and $u(x) = -u(-x)$ for $x > 0$. Then $u \in L^1(R) \cap L^\infty(R)$, $(u^2)_x \in L^1(R)$ but $\int_R |u + \lambda(u^2)_x|\, dx < \int_R |u|\, dx$ for $0 < \lambda < 1$.

EXAMPLE 3.3. Let $\phi(u) = u^2$. Then the closure A of A_1, where $D(A_1) = C_0^1(R)$, $A_1 u = (u^2)_x$ does not satisfy $R(I + A_1) = L^1(R)$. To see this, let $u(x) = \max(x+1, 0)$ for $x < 0$, $u(x) = -u(-x)$ for $x \geqq 0$. Then check directly that $\int_{R^n} |u - v + \lambda(u^2)_x - A_1 v|\, dx \geqq \int_{R^n} |u - v|\, dx$ for $\lambda \geqq 0$ and $v \in D(A_1)$. Assume $v + Av = u + (u^2)_x$. Then there exists $\{u_l\} \subseteq D(A_1)$ such that $u_l + A_1 u_l \to u + (u^2)_x$ in $L^1(R)$. By the above remark, this implies $u_l \to u$ in $L^1(R)$ and $A_1 u_l = (u_l^2)_x \to (u^2)_x$ in $L^1(R)$. Then $u_l{}^2 \to u^2$ *uniformly*. However, $u_l \to u$ in $L^1(R)$ implies (since each u_l is continuous) that there exists a sequence $\{x_l\}$ such that $x_l \to 0$ and $u_l(x_l) = 0$. Then the uniform convergence $u_l{}^2 \to u^2$ implies $u^2(0) = 0$, a contradiction.

The theory of Section 1 and 2 would be much cleaner if, for the A of Theorem 1.1, $v \in D(A)$ implied $\phi(v) \in L^1(R^n)$. This is not the case.

EXAMPLE 3.4. Let $\phi(s) = s^{\frac{1}{3}}$. Let

$$u(x) = \begin{cases} x^{-2}, & x \geqq 1 \\ \max(0, x), & x \leqq 1. \end{cases}$$

Then $u \in L^1(R)$ and $\phi(u)_x \in L^1(R)$. However, $\phi(u(x)) = x^{-2/3}$ for $x > 1$, so $\phi(u) \notin L^1(R)$. Finally, $u \in D(A)$ by Theorem 3.1.

Similarly, it would be nice if $v \in D(A)$ implies $\phi(v) \in L^1_{loc}$. This seems unlikely for $n > 1$.

REMARKS. Most of the examples in this section were derived in conversations with H. Brezis. It should be pointed out that Theorem 3.1 and the remarks following it lead to an existence theorem for (CP) if $n = 1$ which is new and goes beyond the results of Section 2. See [7].

Added in proof. The author is indebted to Ph. Bénilan for pointing out a mistake in a previous formulation of Theorem 3.1. Bénilan has also shown that Theorems 1.1 and 1.2 remain true when the assumption on ϕ iss weakened to $\lim_{r \to 0} \phi(r)/|r|^{(n-1)/n} = 0$ provided that Definition 1.1 is altered slightly.

4. Comments

We begin with remarks about Lemma 2.1 and the proof of Lemma 2.2. Let X be a real Banach space and X^* its dual. Define

$$H(x) = \{x^* \in X^*: (x, x^*) = \|x\| \text{ and } \|x^*\| \leqq 1\}, \tag{4.1}$$

where (x, x^*) denotes the value of x^* at x. The multivalued operator $x \to F(x) = \|x\| H(x)$ is called the *duality map* of X in the semigroup theory. For $X = L^1(R^n)$ and $u \in L^1(R^n)$, $H(u) = \operatorname{sign} u$. Lemma 2.1 is a special case of the following result of Kato [11, Lemma 1.1].

LEMMA 4.1. *Let x, $y \in X$. Then $\|x + \lambda y\| \geqq \|x\|$ for $\lambda \geqq 0$ if and only if there is an $x^* \in H(x)$ such that $(y, x^*) \geqq 0$.*

The lemma clearly remains true if $H(x)$ is replaced by $F(x)$. The proof of sufficiency is essentially the same as in Lemma 2.1. Moreover, the assumption of Lemma 2.1 is seen to be a necessary condition for the conclusion to hold. Lemma 2.2 is a special case of:

LEMMA 4.2. *Let X be a separable Banach space, $\{x_k\} \subseteq X$ and $x_k^* \in H(x_k)$.*

If $\{x_k\}$ converges in X to x, then $\{x_k^*\}$ has a subsequence $\{x_{k_i}^*\}$ convergent weakly-star to an element x^* of $H(x)$.

PROOF. Since $\{x_k^*\}$ is precompact in the weak-star topology on X^*, which is metric on bounded sets since X is separable, there is a weakly-star convergent subsequence $\{x_{k_i}^*\}$ of $\{x_k^*\}$, say $x_{k_i}^* \to x^*$. Then $\|x\| = \lim_{k\to\infty} \|x_k\| = \lim_{k\to\infty} (x_{k_i}, x_{k_i}^*) = (x, x^*)$ and $\|x^*\| \leqq \liminf_{i\to\infty} \|x_{k_i}^*\| \leqq 1$ shows $x^* \in H(x)$. The proof is complete. (Of course, a similar result holds if X is not separable.)

The technical difficultes requiring our precision concerning "sign" in the generality we treated (DE) do not occur in [13], since there only expressions like $(\mathrm{sign}(u-h))f(u,h)$ occur where $f(k,k)=0$. Thus the definition of "sign $(u-h)$" for $u=h$ is immaterial. However, this point does arise in the generality of [13] and is ignored. In particular, concern over this point dictated the order of several limiting procedures in Section 2. The result of H. Brezis, appearing in the appendix to this paper, has clarified this situation.

The generation theorem stated in Section 1 is a special case of the main result of [8]. For example, if A is as in Theorem 1.1, f: $R^n \times R \to R$ has the properties $f(x,0) \in L^1(R^n)$ and $f(x,u)$ is Lipschitz continuous in u uniformly in x, then a semigroup is associated with $A_1 u = Au + f(x,u)$ for $u \in D(A)$. This corresponds to the generalization

$$u_t + \phi(u)_x + f(x,u) = 0$$

of (DE). The results of [9] allow a certain restricted t-dependence $\phi(t,u)$. The most difficult generalization is to an x-dependence, $\phi(x,u)$. Here our type of argument becomes messier. Assuming, e.g., $\phi(x,u)$ is independent of x for $|x|$ large, variants may be carried through. Then the hyperbolic character of (CP) (in particular, bounded domains of dependence), as strongly reflected in the uniqueness theorem of [13], can be used to obtain results. Overcoming the technical difficulties associated with x dependence is one of the main points of [13].

It was mentioned that the allowability of multivalued operators in the Generation Theorem is a convenience. To underscore this, recall that we showed $A_0 v$ is singlevalued on $v \in D(A_0) \cap L^\infty(R^n)$, but did not show it for $v \in D(A_0)$. Even if this were true (and it probably is), a separate argument would be needed to treat the closure A of A_0. In any case, even if A is singlevalued, it would avail nothing to prove it. Single-valuedness has *no known consequence* for the Generation Theorem in nonreflexive spaces.

Finally we remark on the continuity of $S(t)u_0$ in t. It is known that $S(t)u_0$ is

Lipschitz continuous in t (as an $L^1(R^n)$ valued function) for $u_0 \in D(A)$. See [8]. If $\phi \in C^1$, this is the case for $u_0 \in C_0^1(R^n)$. An extended set $\hat{D}(A) \supseteq D(A)$ is defined in [5] such that $S(t)u_0$ is Lipschitz in t precisely for $u_0 \in \hat{D}(A)$. We note that if $\int_{R^n} h(x+\Delta x) - h(x)| \, dx \leqq K|\Delta x|$, $\phi \in C^1$ and ϕ' is bounded, then $h \in \hat{D}(A)$, and the same is true if $\phi \in C^1$ and $h \in L^\infty(R^n)$ as well. To see this, it is enough to note that for h as above and $h \in L^1(R^n) \cap L^\infty(R^n)$

$$u_\varepsilon + \lambda\phi(u_\varepsilon)_x - \varepsilon\Delta u_\varepsilon = h$$

implies

$$\int_{R^n} |u_\varepsilon(x+\Delta x) - u_\varepsilon(x)| \, dx \leqq K|\Delta x|$$

by Proposition 2.3. Hence $\| u_{\varepsilon x_i} \|_1 \leqq K$ for every i. Moreover, $\| \phi(u_\varepsilon)_x \|_1 \leqq K_1 K$ where K_1 is a bound on $|\phi'(s)|$ for $|s| \leqq \| h \|_\infty$. Then limit arguments and properties of $\hat{D}(A)$ given in [5] are used to extend to general h as above. This continuity of $S(t)u_0$ is noted in [3].

Appendix

The results presented in this appendix are due to H. Brezis. Let

$$\operatorname{Sign} r = \begin{cases} \{-1\} & \text{if } r < 0 \\ [-1, 1] & \text{if } r = 0 \\ \{1\} & \text{if } r > 0. \end{cases}$$

The inequality (1.7) in Definition 1.1 involves $\operatorname{sign}_0(v(x) - k)$, which is a particular choice of a function $\alpha \in L^\infty(R^n \times R)$ such that

$$\alpha(x, k) \in \operatorname{Sign}(v(x) - k) \tag{A.1}$$

for almost all $(x, k) \in R^n \times R$. The proof of Proposition 2.1 uses the facts that this choice is independent of v (in the sense that $u(x_0) = v(x_0)$ implies $\operatorname{sign}_0(u(x_0) - k) = \operatorname{sign}_0(v(x_0) - k)$) and the function f in (1.7). Lemma A of this appendix provides two equivalent definitions of A_0, one of which states that in (1.7) $\operatorname{sign}_0(v(x) - k)$ may be replaced by any $\alpha \in L^\infty(R^n \times R)$ satisfting (A.1), and that α may even depend on f.

Let P be the set of all functions $p: R \to R$ which satisfy

(i) p is nondecreasing and Lipschitz continuous and

(ii) p' has compact support.

Let $\tilde{P}$ be the set of $p \in P$ such that $p(+\infty) + p(-\infty) = 0$.

LEMMA A. *Let ϕ be continuous. If v, $w \in L^1(R^n)$ and $\phi(v) \in L^1(R^n)$, then the following assertions are equivalent:*

(i) $w \in A_0 v$.

(ii) *If $p \in P, f \in C_0^\infty(R^n)$ and $f \geqq 0$, then*

$$\text{(A.2)} \qquad \begin{aligned} &- \tfrac{1}{2}(p(+\infty) + p(-\infty)) \int_{R^n} [\phi(v)f_x + wf]dx \\ &+ \int_{R^n} \{p(v)[\phi(v)f_x + wf] - f_x \int_0^v \phi(s)p'(s)ds\}\, dx \geqq 0. \end{aligned}$$

(iii) *If $f \in C_0^\infty(R^n)$ and $f \geqq 0$, then there exists $\alpha \in L^\infty(R^n \times R)$ such that $\alpha(x,k) \in$ Sign$(v(x) - k)$ a.e. in $R^n \times R$ and*

$$\int_{R^n} \alpha(x,k)\{(\phi(v) - \phi(k))f_x + wf\}dx \geqq 0$$

for almost all $k \in R$.

PROOF. It is clear from the definition of A_0 that (i) $\Rightarrow$ (iii). We shall first prove (ii) $\Rightarrow$ (i). Since $f \in C_0^\infty(R^n)$, the expression $\int_0^v \phi(s)p'(s)ds$ in (A.2) can be replaced by $\int_\rho^v \phi(s)p'(s)ds$ for $\rho \in R$. Set $\rho = k$ and $p(s) = \Phi_l'(s-k)$ in (A.2) where Φ_l is given by (1.10). Since $p \in \tilde{P}$,

$$\lim_{l \to \infty} \int_k^{v(x)} \Phi_l'(s-k)\phi(s)ds = \mathrm{sign}_0(v(x) - k)\phi(k),$$

and $\lim_{l\to\infty} \Phi_l'(s-k) = \mathrm{sign}_0(s-k)$, passing to the limit as $l \to \infty$ in (A.2) yields

$$\int_{R^n} \mathrm{sign}_0(v(x) - k)[(\phi(v) - \phi(k))f_x + wf]dx \geqq 0,$$

which implies $w \in A_0 v$.

The proof is completed by showing that (iii) $\Rightarrow$ (ii). Let $f \in C_0^\infty(R^n)$, $f \geqq 0$ and $\alpha \in L^\infty(R^n \times R)$ have the properties stated in (iii).

It is easy to see that if

$$g(k) = \int_{R^n} \alpha(x,k)\{(\phi(v) - \phi(k))f_x + wf\}dx,$$

then $g \in L^\infty(R^n)$. Let $p \in P$ and consider $2K = \int_{-\infty}^{\infty} p'(k)g(k)dk$, which is well defined since p' has compact support. Moreover $K \geqq 0$ since $p' \geqq 0$ and $g \geqq 0$ a.e. by assumption. Now, if $h = \phi(v)f_x + wf$ we have

$$\tfrac{1}{2}\int_{-\infty}^{\infty} p'(k)\left(\int_{R^n}\alpha(x,k)h(x)dx\right)dk = \tfrac{1}{2}\int_{R^n} h(x)\left(\int_{-\infty}^{\infty} p'(k)\alpha(x,k)dk\right)dx$$

$$= \tfrac{1}{2}\int_{R^n} h(x)\left(\int_{-\infty}^{v(x)} p'(k)\alpha(x,k)dk + \int_{v(x)}^{\infty} p'(k)\alpha(x,k)dk\right)dx$$

$$= \tfrac{1}{2}\int_{R^n} h(x)\left(\int_{-\infty}^{v(x)} p'(k)dk - \int_{v(x)}^{\infty} p'(k)dk\right)dx$$

$$= \tfrac{1}{2}\int_{R^n} h(x)[2p(v(x)) - (p(+\infty) + p(-\infty))dx$$

$$= -\tfrac{1}{2}(p(+\infty) + p(-\infty))\int_{R^n} h(x)dx + \int_{R^n} p(v(x))h(x)dx.$$

Similarly,

$$\tfrac{1}{2}\int_{-\infty}^{\infty} p'(k)\left(\int_{R^n}\phi(k)\alpha(x,k)f_x dx\right)dk = \tfrac{1}{2}\int_{R^n} f_x\left(\int_{-\infty}^{\infty} p'(k)\phi(k)\alpha(x,k)dk\right)dx$$

$$= \tfrac{1}{2}\int_{R^n} f_x\left(\left(\int_{-\infty}^{0} + 2\int_0^v - \int_0^{\infty}\right)p'(k)\phi(k)dk)dx\right) = \int_{R^n} f_x\left(\int_0^v p'(s)\phi(s)ds\right)dx.$$

Thus

$$K = -\tfrac{1}{2}(p(+\infty) + p(-\infty))\int_{R^n}[\phi(v)f_x + wf]dx + \int_{R^n}\{p(v)[\phi(v)f_x + fw]$$

$$-f_x\int_0^v \phi(s)p'(s)ds\}\,dx \geqq 0.$$

The proof is complete.

REMARK. It is clear from the proof that if (1) holds for every $p \in \tilde{P}$ (or even for all $\Phi_l'(s-k)$), then (i) holds. Also, if the definition of A_0 is altered by substituting "almost all $k \in R$" for "every $k \in R$" and "$\phi(v) \in L^1_{loc}(R^n)$" for "$\phi(v) \in L^1(R^n)$" then Lemma A remains true if $\phi \in L^1_{loc}(R)$.

In general, A_0 is not closed. However, if ϕ is continuous and there is a constant K such that $|\phi(r)| \leqq K|r|$ for $r \in R$, then A_0 is closed. This is not obvious from Definition 1.1 alone, and we prove it here to illustrate the nature of Lemma A. Indeed, under these assumptions on ϕ, the maps $v \to \phi(v)$, $v \to p(v)\phi(v)$ and $v \to \int_0^v \phi(s)p'(s)ds$ are each continuous mappings of $L^1(R^n)$ into $L^1(R^n)$ for $p \in P$. Hence the set of $[v, w] \in L^1(R^n) \times L^1(R^n)$ for which (A.2) holds is closed, and the result follows immediately.

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